SPACE CHARGE DOMINATED BEAMS

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Canonical description of transverse beam dynamics

Transverse phase space coordinates

\[ x = (x, p_x, y, p_y)^T \]

Dimensionless Hamiltonian

\[
H = \frac{1}{2}(p_x^2 + p_y^2) + L(p_x y - p_y x) + \frac{1}{2}(K_1 x^2 + K_2 y^2) + Gxy + \phi^s
\]

\[ \phi^s(x, y, s) = q \phi / (m \gamma^3 \beta^2 c^2) \]

\( L, K_1, K_2 \) and \( G \) specify the external fields.

For Drift \( L = K_1 = K_2 = G = 0 \)

For Solenoid \( G = 0 \)

\[ K_1 = K_2 = (qB_s / (2m \gamma \beta c))^2 \]

\[ L = qB_s / (2m \gamma \beta c) \]

For normal QP \( L = G = 0 \)

\[ K_1 = -K_2 = (qg / (2m \gamma \beta c)) \]

For skew QP \( L = K_1 = K_2 = 0 \)

\[ G = (qg / (2m \gamma \beta c)) \]
Paraxial equation of motion

\[ x = (x, p_x, y, p_y)^T \]

\[ x'(s) = F(s)x(s) \]

\[ H = \frac{1}{2} (p_x^2 + p_y^2) + L(p_x y - p_y x) \]
\[ + \frac{1}{2} (K_1 x^2 + K_2 y^2) + G_{xy} + \phi^s \]

\[ x'(s) = \frac{dx}{ds} = \frac{\partial H}{\partial p_x} \]
\[ y'(s) = \frac{dy}{ds} = \frac{\partial H}{\partial p_y} \]

\[ p'_x(s) = \frac{dp_x}{ds} = -\frac{\partial H}{\partial x} \]
\[ p'_y(s) = \frac{dp_y}{ds} = -\frac{\partial H}{\partial y} \]

4x4 matrix F(s) specifies the external and self forces acting on a particle. We need to know \( \phi^s \)
Space charge potential

For self-consistent dynamics of beam particles in the transverse phase space we need to solve Vlasov-Maxwell equations for a distribution function $f(x, p_x, y, p_y, s)$ & self-field potential $\phi^s(x, y, s)$

$$\frac{\partial f}{\partial s} + \frac{\partial H}{\partial p} \cdot \frac{\partial f}{\partial x} - \frac{\partial H}{\partial x} \cdot \frac{\partial f}{\partial p} = 0$$

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) \phi^s(x, y, s) = -\frac{q^2 n(x, y, s)}{\varepsilon_0 m \gamma^3 \beta^2 c^2}$$

$$n(x, y, s) = \int f \, dp_x \, dp_y$$
Space charge potential

Uncoupled system (Drift, Quadrupole lattice, Solenoid in LF)

There exists an exact solution of V-M equations (KV)

\[ \phi^S(x, y, s) = -\frac{Q}{(r_x + r_y)} \left( \frac{x^2}{r_x} + \frac{y^2}{r_y} \right) \]

\[ r_x = 2\sqrt{\langle x^2 \rangle} \quad r_y = 2\sqrt{\langle y^2 \rangle} \]

\[ Q = \frac{qI}{(2\pi\varepsilon_0 mc^3 \beta^3 \gamma^3)} \]

\[ I = qn_0 \pi \tilde{r}_x \tilde{r}_y \beta c \]

Beam density in the transverse configuration space remains constant

Pulsating elliptical cross-section of the beam remains upright during transport
**Space charge potential**

**Coupled system** (solenoid, spiral inflector, Skew quadrupole)

The elliptical cross-section of the beam not only pulsates but also rotates about the propagation axis.

We need transformation of coordinates where elliptical cross-section of the beam in real space is upright.

$$H = \frac{1}{2} (p_x^2 + p_y^2) + L(p_x y - p_y x) + \frac{1}{2} (K_1 x^2 + K_2 y^2) + G_{xy} + \phi^s$$
Space charge potential

\[ \phi^S(x, y, s) = -\frac{Q}{2} (\phi_{xx} x^2 - 2\phi_{xy} xy + \phi_{yy} y^2) \]

\[ \phi_{xx} = \frac{\tilde{r}_x + \tilde{r}_y - (\tilde{r}_x - \tilde{r}_y) \cos 2\theta}{\tilde{r}_x \tilde{r}_y (\tilde{r}_x + \tilde{r}_y)} \]

\[ \phi_{yy} = \frac{\tilde{r}_x + \tilde{r}_y + (\tilde{r}_x - \tilde{r}_y) \cos 2\theta}{\tilde{r}_x \tilde{r}_y (\tilde{r}_x + \tilde{r}_y)} \]

\[ \tilde{r}_x = \frac{1}{\sqrt{2}} \sqrt{\sigma_{11} + \sigma_{33} + \sqrt{[\sigma_{11} - \sigma_{33}]^2 + 4\sigma_{13}^2}} \]

\[ \theta = \frac{1}{2} \tan^{-1} \left( \frac{2\sigma_{13}}{\sigma_{11} - \sigma_{33}} \right) \]

\[ \tilde{r}_y = \frac{1}{\sqrt{2}} \sqrt{\sigma_{11} + \sigma_{33} - \sqrt{[\sigma_{11} - \sigma_{33}]^2 + 4\sigma_{13}^2}} \]

4D hyper-ellipsoid

\[ x^T \sigma^{-1} x = 1 \]
Determination of the transverse beam sizes and projected emittances

Infinitesimal transfer matrix:

\[ x'(s) = F(s)x(s) \]

\[ x(s_0) = (x, p_x, y, p_y)^T \]

\[ x(s) = M(s, s_0)x(s_0) \]

Solution can be a linear combination of four linearly independent solutions

\[ \sigma(s) = M(s, s_0)\sigma(s_0)M(s, s_0)^T \]

To generate the matrix \( M(s, s_0) \), we need to solve above equation for four different initial conditions at interval \( ds = s-s_0 \)
Beam Envelope

Moment method:

Propagating the second moments of the beam distribution instead of solving the paraxial equations of motion

\[ x'(s) = F(s)x(s) \]

\[ \sigma'(s) = F(s)\sigma(s) + \sigma(s)F^T(s) \]

Coupled system: 10 independent first order differential equations for sigma matrix elements

Uncoupled system: 6 independent equations
Beam size and Emittance

\[
\begin{align*}
  r_x(s) &= \sqrt{\sigma_{11}(s)} \\
  r_y(s) &= \sqrt{\sigma_{33}(s)}
\end{align*}
\]

\[
\sigma(s) = M(s, s_0)\sigma(s_0)M(s, s_0)^T
\]

\[
\sigma'(s) = F(s)\sigma(s) + \sigma(s)F^T(s)
\]

Due to coupling of transverse motions, projected emittances are not conserved in the lab frame.

Intrinsic degradation of the full 4D phase space

\[
\begin{align*}
  \varepsilon_x(s) &= \sqrt{[\sigma_{11}(s)\sigma_{22}(s) - \sigma_{12}^2(s)]} \\
  \varepsilon_y(s) &= \sqrt{[\sigma_{33}(s)\sigma_{44}(s) - \sigma_{34}^2(s)]} \\
  \varepsilon_g^2 &= \frac{1}{2}(\varepsilon_x^2 + \varepsilon_y^2) + \sigma_{13}\sigma_{24} - \sigma_{14}\sigma_{23}
\end{align*}
\]
Steps involved

- Find field components and get Hamiltonian $H$
- Obtain paraxial equations of motion
  \[ x'(s) = F(s)x(s) \]
- Obtain space charge potential $\phi^S(x, y, s)$
- Solve equations of sigma matrix at small steps
  \[ \sigma(s) = M(s, s_0)\sigma(s_0)M(s, s_0)^T \]
  \[ \sigma'(s) = F(s)\sigma(s) + \sigma(s)F^T(s) \]
- Calculate evolution of beam size and emittance from elements of sigma matrix.
Hamiltonians

Drift

\[ H = \frac{1}{2} (p_x^2 + p_y^2) + \phi^s \]

Quadrupole

\[ H = \frac{1}{2} (p_x^2 + p_y^2) + \frac{1}{2} K(x^2 - y^2) + \phi^s \]

Skew Quadrupole

\[ H = \frac{1}{2} (p_x^2 + p_y^2) + G_{xy} + \phi^s \]

Solenoid

\[ H = \frac{1}{2} (p_x^2 + p_y^2) + L(p_x y - p_y x) + \frac{1}{2} K(x^2 + y^2) + \phi^s \]

Solenoid in Larmor Frame

\[ H = \frac{1}{2} (p_x^2 + p_y^2) + \frac{1}{2} K(x^2 + y^2) + \phi^s \]
Example Applications
Self-consistent space charge dominated beam dynamics in a spiral inflector

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Spiral Inflector

Hamiltonian in self-field and applied electric and magnetic fields

\[ H = \frac{1}{2} \left[ \left( p_u - \frac{FC}{A} h \right)^2 + \left( p_h + \frac{FC}{A} u \right)^2 + \left( p_v + \frac{2}{A} u - \frac{2FS}{A} h \right)^2 \right] \]

\[ - \frac{\xi}{2A^2} (u - k'Sh)^2 - \frac{u^2}{2A^2} - \frac{Kk'}{A^2} (C^2u^2 + h^2) + \frac{FS}{A^2} uh + \phi^{sc} \]

\[ \mathbf{x}'(s) = \mathbf{F}(s)\mathbf{x}(s) \]

\[ \xi = \frac{1 + 2Kk'S^2}{1 + k'^2S^2} \]

\[ C = \cos(s/A) \]

\[ S = \sin(s/A) \]

\[ F = K + \frac{k'}{2} \]

\[ \mathbf{\sigma}'(s) = \mathbf{F}(s)\mathbf{\sigma}(s) + \mathbf{\sigma}(s)\mathbf{F}^T(s) \]

NIM A 693 (2012) 276
Converging non-axi-symmetric beam with equal emittances at input in both the planes causes less emittance growth at the exit.
Beam Inflection using Spiral Inflector

Transported 1 mA, an elliptical solenoid is needed
Self-consistent simulation of the space charge dominated beams in an elliptical solenoid magnet

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\[
A_x(s) = -\frac{1}{2}B_z(s)y, \quad A_y(s) = \frac{1}{2}B_z(s)x \quad \text{and} \quad A_z(s) = \frac{1}{2}D(s)xy
\]

\[
H = -\frac{qA_x}{P} + \frac{1}{2}\left[\left(p_x - \frac{qA_x}{P}\right)^2 + \left(p_y - \frac{qA_y}{P}\right)^2\right] + \phi^S
\]

\[
H = -J(s)xy + \frac{1}{2}\left[(p_x + K(s)y)^2 + (p_y - K(s)x)^2\right] + \phi^S
\]

\[
K(s) = qB_z(s)/(2m\gamma\beta c) \quad \text{and} \quad J(s) = qD(s)/(2m\gamma\beta c).
\]
Elliptical Solenoid

$r_x(0) = r_y(0) = 0.25 \text{ cm}$ 

$I = 10 \text{ mA}$

$\sigma (\text{slit}) = 135 \text{ cm}$

$s (\text{cm})$

$r_x(\text{cm})$

$r_y(\text{cm})$

$I = 10 \text{ mA}$

$K = 0.04 \text{ cm}^{-1}$

$s (\text{cm})$

$\xi, \eta$

$\xi_x, \xi_y$

$\eta_x, \eta_y$
Elliptical Solenoid

\[ X(0) = Y(0) = 0.25 \text{ cm} \quad \zeta_X(0) = \zeta_Y(0) = 60 \text{ mm rad} \]

\[ I = 5 \text{ mA} \]

\[ s \text{ (cm)} \]

\[ X \text{ (cm)} \]

\[ Y \text{ (cm)} \]

\[ \theta \text{ (mrad)} \]

\[ X \text{ - plane} \]

\[ Y \text{ - plane} \]

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A Vlasov equilibrium for space charge dominated beam in a misaligned solenoidal channel

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The effect of displacement and rotational misalignments of solenoid magnets with respect to the ideal beam propagation axis on the dynamics of intense charged particle beams have been studied. The equation of motion of the beam centroid has been obtained and found to be independent of any specific beam distribution. A Vlasov equilibrium distribution for the intense beam propagation through misaligned focussing channel has been obtained. Self-consistent simulation confirms the analytical result. © 2012 American Institute of Physics. [http://dx.doi.org/10.1063/1.4747694]
Vlasov Equilibrium in misaligned Solenoid lattice

- Envelope of high current beam is independent of centroid motion.
- Envelope remains stable while centroid may not.

Kinetic equilibrium of space charge dominated beams in a misaligned quadrupole focusing channel

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The dynamics of intense beam propagation through the misaligned quadrupole focusing channel has been studied in a self-consistent manner using nonlinear Vlasov-Maxwell equations. The equations of motion of the beam centroid have been developed and found to be independent of any specific beam distribution. A Vlasov equilibrium distribution and beam envelope equations have been obtained, which provide us a theoretical tool to investigate the dynamics of intense beam propagating in a misaligned quadrupole focusing channel. It is shown that the displaced quadrupoles only cause the centroid of the beam to wander off axis. The beam envelope around the centroid obeys the familiar Kapchinskij-Vladimirskij envelope equation that is independent of the centroid motion. However, the rotation of the quadrupole about its optical axis affects the beam envelope and causes an increase in the projected emittances in the two transverse planes due to the inter-plane coupling. © 2013 AIP Publishing LLC. [http://dx.doi.org/10.1063/1.4813251]
Kinetic Equilibrium in misaligned quadrupole lattice

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Effect of subdominant species on the evolution of intense primary beam in a low energy beam transport line

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The dynamics of space-charge-dominated multispecies beam with different distributions has been studied in a solenoid based low energy beam transport line using a particle-in-cell simulation method. Results are presented in wide parameter regimes covering emittance dominated as well as space charge dominated multispecies beams consisting of \( p \), \( H_2^+ \), and \( H_3^+ \) beams. Simulation shows the formation of hollow distribution of \( H_2^+ \) and \( H_3^+ \) beams around the desired proton beam and separation is more distinct as the fraction of proton is increased. The hollow formation is almost independent of combination of \( H_2^+ \) and \( H_3^+ \) beams, once the fraction of proton beam is fixed. The emittance growth of proton is found to increase sharply in the space-charge-dominated regime and is more in the cases where proton fraction is low. © 2013 AIP Publishing LLC. [http://dx.doi.org/10.1063/1.4817952]
Beam Transport

K-V Beam envelope equations

K-V envelope equations with linear space charge

\[
x''_m + k_x(s)x_m - \frac{2K}{x_m + y_m} - \frac{\varepsilon_x^2}{x_m^3} = 0
\]

\[
y''_m + k_y(s)y_m - \frac{2K}{x_m + y_m} - \frac{\varepsilon_y^2}{y_m^3} = 0
\]

K=generalized perveance

\[
K = \frac{q}{4\pi m \varepsilon_0 c^3} \cdot \frac{2I}{\beta^3 \gamma^3} = \frac{1}{I_0} \cdot \frac{2I}{\beta^3 \gamma^3}
\]

\[
I_0 = \frac{A}{Q} \cdot \pi \cdot 10^7 A
\]

Only stationary distribution function first developed in 1959.

➢ Plays an important role in high intensity beam physics.

➢ Valid for continuous uniform density beam within elliptical cross-section.

➢ Valid for only uncoupled lattice.

What if distribution function is other than K-V ????
Application 1: Limiting current

Transverse space charge effect increases the beam size and responsible for axial beam loss (serious at low energies).

\[ I_m = \frac{I_0}{2} \beta^2 \gamma^2 \varepsilon_n \frac{\nu_0}{R} \cdot \frac{\Delta \phi}{2\pi} \]

⇒ Possible solutions:

a. inject beam at high energy (~100keV)

b. use large injection radius

c. provide sufficient axial focusing

d. use high energy gain per turn
Application 1: Limiting current

\[ \nu_{y_{\text{max}}}^2 = -\beta^2 \gamma^2 + \frac{N^2}{N^2 - 1} \left( \frac{B_H - B_V}{4B_H B_V} \right)^2 \]

\[ I_{\text{max}} = \frac{I_0 \beta^3 \gamma^3 \Delta \phi}{2} \times \frac{q^2 a^2 (B_H + B_V)^2}{4m^2 c^2 \beta^2 \gamma^2} \left( -\beta^2 \gamma^2 + \frac{N^2}{N^2 - 1} \left( \frac{B_H - B_V}{(B_H + B_V)^2} \right)^2 \right) - \frac{\varepsilon_n^2}{a^2} \]
Application 2: Central region acceptance

Hills & valleys are treated as bending magnets.
For flaring & edge effect we used thin lenses at H-V boundary.
Four gaps with 100 kV each...

\[ X'' + k^2 X - \frac{4I}{(X+Y)I_0 \beta^3 \gamma^3} \cdot \frac{2\pi}{\Delta \phi} - \frac{\varepsilon_x^2}{X^3} = 0 \]

\[ Y'' - \frac{4I}{(X+Y)I_0 \beta^3 \gamma^3} \cdot \frac{2\pi}{\Delta \phi} - \frac{\varepsilon_y^2}{Y^3} = 0 \]

\[ \beta_x = \frac{X^2}{\varepsilon_x}, \quad \alpha_x = -\frac{XX'}{\varepsilon_x}, \quad \gamma_x = \frac{1+\alpha_x^2}{\beta_x} \]

\[ J_2 = R \cdot J_1 \cdot R^{-1}, \quad J = \begin{bmatrix} \alpha & \beta \\ -\alpha & -\gamma \end{bmatrix} \]

\[ \varepsilon_n (= \beta \gamma \cdot \varepsilon_x = \beta \gamma \cdot \varepsilon_y) = 0.8 \pi \text{ mmrad} \]
Transverse beam envelopes with uniform cylindrical bunch in a compact cyclotron

A. Goswami, P.S. Babu V.S. Pandit  NIM A 562 (2006)34
Application 3: Accelerated beam envelopes

Transverse beam envelopes with uniform ellipsoidal bunch in a compact cyclotron

\[
X'' + \frac{(\beta \gamma)'}{(\beta \gamma)} X' + \left[ \frac{\nu_x^2}{R^2} - \frac{3 I c}{2 I_0 \beta^2 \gamma^3 f_{rf}} \frac{1}{X^3} G \left( \frac{Y}{X}, \frac{Z}{X} \right) \right] - \frac{\epsilon_{nx}^2}{\beta^2 \gamma^2 X^3} = 0
\]

\[
Y'' + \frac{(\beta \gamma)'}{(\beta \gamma)} Y' + \left[ \frac{\nu_y^2}{R^2} - \frac{3 I c}{2 I_0 \beta^2 \gamma^3 f_{rf}} \frac{1}{Y^3} G \left( \frac{X}{Y}, \frac{Z}{Y} \right) \right] - \frac{\epsilon_{ny}^2}{\beta^2 \gamma^2 Y^3} = 0
\]

Application 4: Ion source and LEBT
The coupled K-V envelope equations in Larmor frame

\[ X'' + k^2 X - \frac{2K}{X + Y} - \frac{\mathcal{E}_n^2}{\beta^2 \gamma^2 X^3} = 0 \]

\[ Y'' + k^2 Y - \frac{2K}{X + Y} - \frac{\mathcal{E}_n^2}{\beta^2 \gamma^2 Y^3} = 0 \]

\[ X(0) = Y(0) = 2.5 \text{mm} \quad \mathcal{E}_n = 0.8 \text{ mm} \]

If beam is initially correlated

???

Then

???
Application 5; Multispecies beam

Beam from MW ion source consists of
\[ p(\sim 80\%), \ H_2^+ (\sim 15\%), \ H_3^+ (\sim 5\%) \]

\[
n_j(r,s) = \begin{cases} 
\frac{N_j}{\pi \cdot r_j^2(s)}, & 0 < r \leq r_j(s) \\
0, & r > r_j(s)
\end{cases}
\]

\[
E_{x_j}(x) = \begin{cases} 
\frac{I_j}{2\pi \varepsilon_0 \beta_j c r_j^2} \cdot x, & |x| \leq r_j \\
\frac{I_j}{2\pi \varepsilon_0 \beta_j c} \cdot \frac{x}{r^2}, & |r| > r_j
\end{cases}
\]

\[
\theta_j(s) = -\int_{s_i}^{s} k l_j(s') \cdot ds'
\]

\[
x'' = -k l_j^2(s) x + a_j \frac{x}{r_j^2} + \sum_{k=1}^{n} b_{jk} \frac{x}{r_k^2} \cdot \Theta(r_k - r) + \sum_{k=1}^{n} b_{jk} \frac{x}{r^2} \cdot \Theta(r - r_k)
\]
\[ r_j'' + kl_j^2(s) \cdot r_j - \frac{a_j}{r_j} = 4 \cdot \sum_{k \neq j}^{n} b_{jk} \cdot f(r_j,r_k) - \frac{4 \cdot \sum_{k \neq j}^{n} b_{jk} \cdot g(r_j,r_k)}{r_j^3} - \varepsilon_j^2(s) = 0 \]

P. Sing Babu, A. Goswami, V. S. Pandit, Physics of Plasmas 18, 103117, (2011).
Thanks